

Triggering synchronized oscillations through arbitrarily weak diversity in close-to-threshold excitable media

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It is shown that an arbitrarily weak (frozen) heterogeneity can induce global synchronized oscillations in excitable media close to threshold. The work is carried out on networks of coupled van der Pol-FitzHugh-Nagumo oscillators. The result is shown to be robust against the presence of internal dynamical noise.

Emergent synchronized oscillations is a key subject in a variety of fields ranging from physics to biology or medical sciences. In the last few years several papers have been published concerned with the possibility of triggering global oscillatory behavior through heterogeneity and/or internal (dynamical) noise in the excitable medium [1–7]. Although these works are a significant step towards the understanding of emergent oscillatory behavior, many points remain unclear. For instance, in [3] a model was proposed to investigate emergent oscillations in pancreatic β cells [8], in which half of the elements was in the silent phase while the other half was continuously active. Although the approach gave clean globally synchronized oscillations, it is doubtful whether such a *symmetric* arrangement may have any biological meaning. The combined effects of diversity [1–3] and internal dynamical noise [5,9,10] have also been investigated [4,6,11,12]. While, a common conclusion seems to be the significant role of dynamical noise in triggering global oscillations, a key point such as the size dependence of the results was not investigated in detail. Here we start from Cartwright approach [3] and explore the possibility of synchronization as a function of the amount of diversity (fraction of diverse elements). We show that, as the system approaches threshold for oscillatory behavior, the number of diverse elements required to trigger global oscillations becomes arbitrarily small. This is particularly appealing from a biological point of view as the possibility of having a small number of cells different from the rest is always there. We also show that these results are not significantly affected by internal dynamical noise.

We base our analysis upon the van der Pol-FitzHugh-Nagumo equations [13–16] that, as discussed in [3], are an adequate mathematical description of the circuit model (which involves a capacitor, a nonlinear resistance across the capacitor, an inductance and a resistance) commonly used to represent a physiological excitable medium. Including first-nearest-neighbors coupling between the ele-

ments in the network [3,17,18] the equations are written as,

$$\dot{\psi}_i = \gamma(\phi_i - \psi_i^3/3 + \psi_i) \quad (1a)$$

$$\dot{\phi}_i = -\gamma^{-1}[\psi_i + (\nu_i + \nu_0\eta(t)) + \beta\phi_i] + \kappa \sum_{j=1}^N (\psi_j - \psi_i). \quad (1b)$$

where N is the number of elements in the network. All constants and variables are dimensionless. Variables ψ and ϕ are proportional to the potential across the nonlinear resistance (cell membrane) and the current through the supply, respectively. The subindex i indicates an element in the network. The constant ν is proportional to the potential supplied, β to the (membrane) resistance and γ to the square root of the quotient inductance/capacitance. The coupling between the elements in the network is accounted for by κ (see [3] for a thorough discussion). In the model we allow constant ν to be different on each element i of the network and to fluctuate dynamically ($\eta(t)$ is a gaussian noise and ν_0 a constant).

In order to quantify the emergence of oscillatory behavior we calculate the spatiotemporal average of variable ψ , namely,

$$\sigma_o = \sqrt{\frac{1}{N(t_f - t_i)} \sum_{j=1}^N \sum_{t=t_i}^{t_f} [\psi_j^2(t) - \langle \psi_j(t) \rangle^2]} \quad (2)$$

where $\langle \psi_j(t) \rangle$ represents the temporal average of the j -th potential. The initial time t_i is chosen so that the contribution of transients to the average is minimized (note that for ν near threshold the transient can be very long, see below) while t_f is taken to cover a sufficiently large number of periods of the system in its oscillatory phase (for the values of the parameters given below the internal period varies around 10).

Synchronization was in its turn evaluated by calculating the following average,

$$\sigma_s = \sqrt{\frac{1}{(N-1)(t_f - t_i)} \sum_{j=2}^N \sum_{t=t_i}^{t_f} [\psi_j(t) - \psi_1(t)]^2} \quad (3)$$

where site "1" was randomly chosen. We could have extended the sum to all pairs of elements $\langle ij \rangle$ but this would have prohibitively increased computation time in large networks. Note that using σ_s to test synchronization is far more demanding than most tests used in previous analyses. In Eqs. (2) and (3) the discrete sum in t clearly accounts for (numerical) time averages.

In the following we take $\beta=0.5$ and $\gamma=2$ and vary the remaining parameters. In particular we investigate, for a given ν , how σ_o and σ_s vary with the fraction x of elements with $-\nu$ (hereafter referred to as impurity elements or, simply, impurities) distributed randomly in the network. In the absence of both internal noise ($\nu_0=0$) and coupling ($\kappa=0$), oscillatory behavior occurs for $|\nu| < \nu_c = 0.60412$ (for the values of β and γ chosen here, see [3]). Calculations were carried out on $L \times L$ clusters ($L = 10 - 40$) with periodic boundary conditions and an integration step $\Delta t = 0.002$.

In Figure 1 we plot the spatial average of ψ for 10×10 networks with $\nu = 0.62$ on all elements but two that have $\nu = -0.62$, with and without coupling (in the former case $\kappa = 0.5$). First we note that, as remarked above, the stationary state in the uncoupled case is only reached after rather long times ($t > 100$). Instead, in the coupled case the transient is very short and the system soon shows a coherent oscillatory behavior. Importantly, the existence of emergent oscillations do not depend on the actual location of impurity elements. In Figure 2 we show the average ψ over the network and over five realizations of quenched disorder (different spatial configuration of impurities). As the period of oscillation is weakly dependent on the location of impurity elements, the resulting pattern is a typical sum of oscillators with slightly different periods. It is interesting to note that even in the case that the two impurities lie at neighboring sites, global persistent oscillations emerge upon coupling. The results of Figure 1 are truly remarkable as global oscillations are promoted by a very weak diversity (2% in this case). Some characteristics of this central result are discussed in detail hereafter.

Figures 3 and 4 show the parameters that characterize the emergence of oscillation and synchronization (σ_o and σ_s), versus the fraction x of impurities, for $\nu = 1.0$ and 0.61 , *i.e.*, far and close to threshold, respectively. The results correspond to networks of linear size $L=20$ and 40 with coupling constant $\kappa=2$ and 8 for $L=20$ and 40 , respectively. This choice was motivated by the scaling argument of Ref. [6] (see also [11] and the discussion below), according to which one obtains solutions with

similar properties in two systems of linear size L and aL if the diffusive coupling constant of the latter is increased by a factor a^2 (apart from border effects). Averages were taken over five realizations (some checks with up to twenty realizations led to similar results) and in the time range $t=200-600$. First we discuss the results without dynamical noise. The critical impurity fraction x_c (value of x at which σ_o steeply increases) for $\nu=0.61$ and 1 approximately lies at $x_c \approx 0.006$ and 0.2 , respectively, the results being almost independent of size, particularly in the former case, although the sharpness of the transition to the oscillatory phase increases with the size of the network, as can be noted in Figure 4. On the other hand, x_c shows no dependence on the coupling constant κ , as indicated by the results of Figure 3 and 4 and other data not shown in the Figures (this is so once κ is beyond a critical value, see [3]). In fact, x_c can be derived, within a more than reasonable approximation, from a simple mean field approach, according to which the onset will take place when $\langle \nu \rangle = (1 - 2x)\nu$ equals ν_c . This leads to $x_c = 0.5(1 - \nu_c/\nu)$, which gives $x_c=0.006$ and 0.198 for $\nu=0.61$ and 1.0 , quite compatible with the numerical results of Figures 3 and 4. Note that the parameter that characterize synchronization σ_s is significantly smaller for $\nu = 0.61$ [19]. A plausible explanation for this behavior is that as for $\nu = 1.0$ the transition occurs at much larger impurity concentrations, clustering is more probable, increasing the difficulty of synchronizing the whole system.

Dynamical noise does not qualitatively change the results discussed above. Figure 3 shows results for $\nu = 0.61$ and $\nu_0 = 1$ [20]. The most noticeable (quantitative) changes are: i) At $x=0$, σ_o is higher than in the absence of dynamical noise, although it is still not sufficiently large so as to consider the system being in its oscillatory phase, ii) consequently, the transition is less sharp, and, iii) synchronization is decreased (larger σ_s). These results are in apparent contradiction with several analyses which indicate that dynamical noise increases oscillation and synchronization [4,6,11]. However, this may be well due to the non-optimal noise level which is required for coherent resonance and to the more severe measures of oscillation and synchronization that we have adopted. Moreover, we note that in those studies nothing was said about whether the effect survives as the size of the system increases. Preliminary results indicate that in fact it does not, in line with the rather small increase of σ_s that dynamical noise promotes near $x=0$ and the decrease in synchronization. In any case the main conclusion of this analysis is that dynamical noise does not modify the previous result, that is, the dramatic effect that a small number of impurities has in systems near threshold.

A final point concerns the effect of the coupling constant. Results for networks of $L=20$ and 40 , $\nu=0.61$ and $x=0.02$, are shown in Figure 5. It is noted that σ_o reaches its maximum (constant) value for a coupling constant

that is significantly lower for the smaller network. In fact this occurs at $\kappa \approx 0.45$ and 1.7 for $L=20$ and 40 , respectively. This is in accordance with the expected behavior (derived from the diffusive character of the coupling term) discussed above. The results for the synchronization parameter are similar: to reach the same small levels of σ_s (less than 0.5 , say) the coupling constant in $L=40$ should be 4 times larger. Despite the usefulness of the scaling trick in numerical computations, one should recall that in realistic systems the coupling between elements is typically intensive, that is independent of the system's size, and it is determined by intrinsic properties. Generally our simulations show that, when the coupling constant is kept fixed, the emergent oscillations and the degree of synchronization are less and less pronounced as the number of constituents is increased. This worsening occurs either when these effects are induced solely by noise and when they are triggered by diversity, as discussed here. As far as experimental results are concerned, the basic point for the synchronous behavior to be observed is the strength of the effective coupling constant with respect to the number of elements. Another important feature is the type of interaction. Indeed, it is possible that non-diffusive couplings may lead more efficient mechanisms of synchronization.

Summarizing, here we have discussed the possibility of triggering global oscillations in close-to-threshold excitable media through an arbitrarily weak heterogeneity. The work was carried out by assuming the existence of two possible types of elements in the network, one silent and another continuously active. The results clearly indicate that when the system is near threshold, global (synchronized) oscillations emerge for a small number of diverse elements. Dynamical noise does not alter this conclusion, although it may introduce some significant changes such as a decrease in synchronization.

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- [19] In absence of synchronization the parameter σ_s is always above 1 (see Fig. 5).
- [20] For these values of the parameters, uncoupled elements do oscillate.

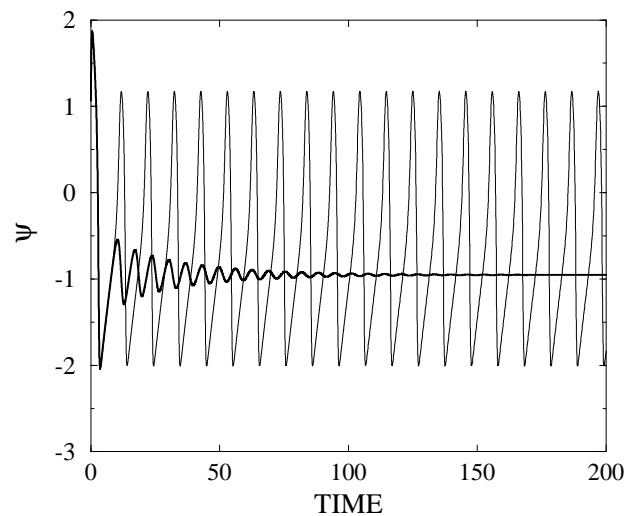


FIG. 1. Variable ψ as a function of time in a 10×10 network of van der Pol-FitzHugh-Nagumo elements with $\beta = 0.5$, $\gamma = 2$ and $\nu = 0.62$ for all elements but two with $\nu = -0.62$ located at random in the network. For those values of β and γ the threshold for oscillatory behavior in an isolated element occurs at $\nu_c = 0.60412$. The results correspond to no coupling between the elements $\kappa = 0.0$ (thick continuous line) and for $\kappa = 0.5$ (thin line). In the latter case the average of ψ over the whole network is shown.

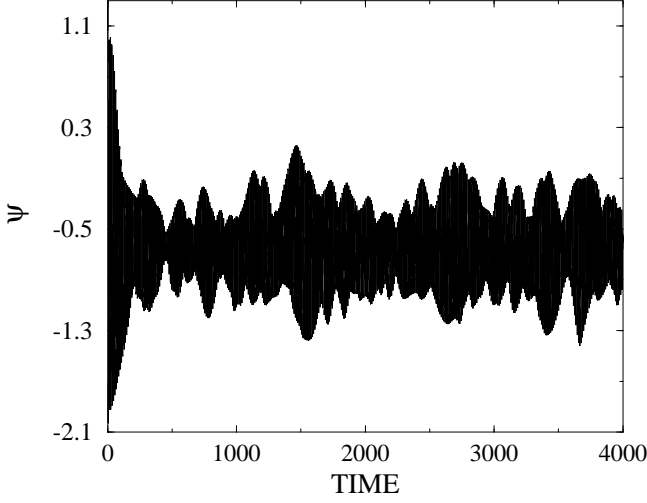


FIG. 2. Variable ψ as a function of time in networks with the same parameters of Fig. 1 and $\kappa = 0.5$. The result represents an average over the whole network and over five realizations (each corresponding to a random spatial distribution of the two elements with $\nu = -0.62$).

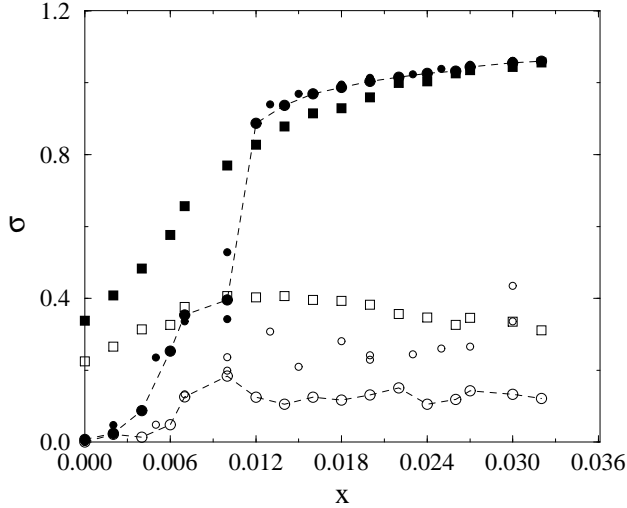


FIG. 3. Filled symbols: Parameter used to quantify the emergence of oscillatory behavior, as defined in Eq. (2), in a heterogeneous excitable media described by Eq. (1) with $\nu = 0.61$ versus the fraction of elements x in the network with $\nu = -0.61$ (averages were done over 5 realizations of the disordered network). The numerical results correspond to networks of size 20×20 and 40×40 (symbol size proportional to the linear size of the network). The rest of the parameters in the van der Pol-FitzHugh-Nagumo medium are: $\beta = 0.5$, $\gamma = 2$, and values of κ discussed in the text. Empty symbols: Same for the parameter used to quantify synchronization, as defined in Eq. (3). Circles: without dynamical noise. Squares: with dynamical noise ($\nu_0 = 1$). The lines are guides to the eye.

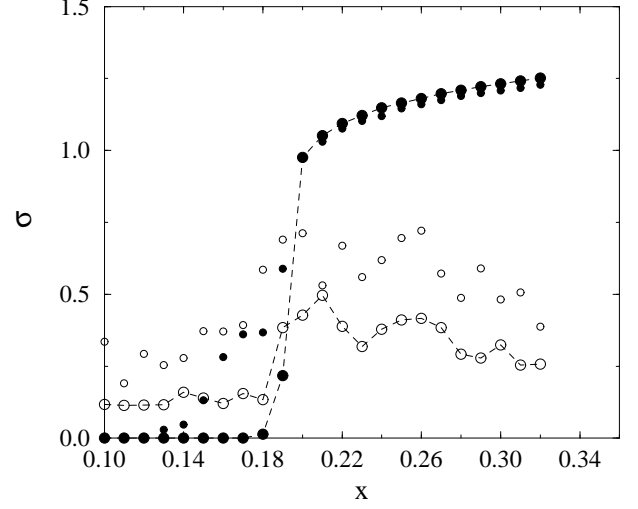


FIG. 4. Same as Figure 3 for $\nu = \pm 1.0$. Only results without dynamical noise are shown.

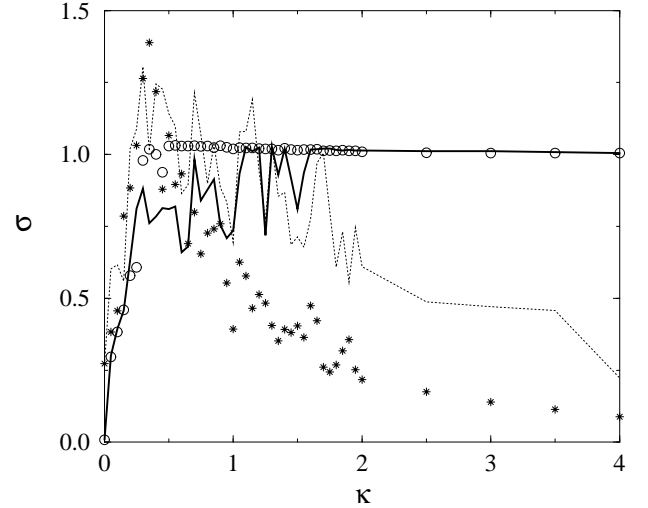


FIG. 5. Parameters used to quantify the emergence of oscillatory behavior (σ_o) and synchronization (σ_s) versus coupling κ . The results correspond to a fraction of impurities of 0.02, $\nu = 0.61$, $\beta=0.5$, $\gamma = 2$ and bidimensional networks of linear size 20 (σ_o empty circles and σ_s stars) and 40 (σ_o thick continuous line and σ_s dotted line). No dynamical noise was included in the calculation. Averages were done over 5 realizations of the disordered network.